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A modified elliptic Lindstedt–Poincaré method for certain strongly non-linear oscillators☆

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Abstract

A modified elliptic Lindstedt–Poincaré method is presented for the steady state analysis of strongly non-linear oscillators of the form $\ddot{x} + c_1 x + c_3 x^3 = \varepsilon f(x, \dot{x})$, in which a new parameter $\alpha = \alpha(\varepsilon)$ is employed such that the value of α is always small regardless of the magnitude of the original parameter ε. Therefore, the above strongly non-linear oscillators with large parameter ε is transformed into a small parameter system with respect to α.

1. Introduction

Over the last century many researchers have devoted their attention to the study of approximate analytical solutions to the non-linear oscillator equation

$$\ddot{x} + x = \varepsilon f(x, \dot{x}), \tag{1}$$

where ε is a small positive parameter and f is a polynomial function of its arguments. The classical techniques for solving Eq. (1) include the L–P method, the Krylov–Bogoliubov–Mitropolsky (KBM) method and the multiple time scales (MTS) method, as described by Nayfeh [1] and Mickens [2]. However, these classical perturbation methods traditionally are restricted to solving problems with weak non-linearity. Mickens and Oyedeji [3] investigated a new class of non-linear

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oscillator equation

$$\ddot{x} + x^3 = \varepsilon f(x, \dot{x}) \tag{2}$$

by using the harmonic balance (HB) method and the slowly varying amplitude and phase method with circular functions. Following the Mickens techniques, Yuste and Bejarano [4] also investigated Eq. (2) but adopted Jacobian elliptic functions instead of circular functions. The accuracy of the elliptic functions method is obviously higher than that of the circular functions method. Recently some researchers presented several techniques for the more general case of the form

$$\ddot{x} + c_1 x + c_3 x^3 = \varepsilon f(x, \dot{x}). \tag{3}$$

For example, Margallo et al. [5,6] presented an elliptic HB method; Yuste and Begarano [7] developed an elliptic Krylov–Bogoliubov (KV) method; and Coppola and Rand [8] used an elliptic averaging method. All the methods mentioned above have their own advantages to obtain approximate analytical solutions. However, most of them only give first order approximate solutions which are expressed as $x = A_0 ep(\omega t, k)$, where A_0, ω, k are constants and ep denotes a convenient Jacobian elliptic functions. These solutions can give very good approximations when ε is a small value. However, when ε is not a small one, they can have large unacceptable errors.

Recently, Chen and Cheung [9] have presented an elliptic perturbation method for studying the oscillator Eq. (3), in which the Jacobian elliptic functions are employed instead of the usual circular functions in the classical L–P perturbation procedure. The method give the periodic solution generally by $x = A_0 ep(\omega t, k), \mathrm{d}\tau/\mathrm{d}t = \omega(\tau) = \omega_0 + \varepsilon\omega_1(\tau)$ and it is better than some other elliptic methods.

Chen and Cheung [10] also presented an elliptic Lindstedt–Poincaré method for studying the oscillator Eq. (3), in which the Jacobian elliptic functions are used instead of the usual circular functions in the classical L–P perturbation procedure. The method gives the periodic solution generally by $x = A_0 ep(\omega t, k) + \varepsilon x_1(\omega t, k)$, $\omega = \omega_0 + \varepsilon\omega_1$ and improves the solution by the first order correction x_1. So it is more accurate than other elliptic methods which give only first order solution. This method gives very good approximate analytical solutions when ε is a small value. However, when ε becomes quite large, the approximations does not agree with those of the R–K method very well.

In this paper, a new modified elliptic perturbation method is presented for studying the oscillator Eq. (3). In this procedure a parameter $\alpha = \alpha(\varepsilon)$ is defined which enables a strongly non-linear system corresponding to the original parameter ε to be transformed into a small parameter system with respect to α. Applying the elliptic Lindstedt–Poincaré method to the new system respect to α, we can improve the accuracy of the solution. In order to assess the applicability of the proposed method, a numerical comparison has been made between the present method, the elliptic Lindstedt–Poincaré method and the R–K integration method.

2. The modified elliptic Lindstedt–Poincaré method

Consider the equation

$$\ddot{x} + c_1 x + c_3 x^3 = \varepsilon f(x, \dot{x}), \tag{4}$$

where dots denote derivatives with respect to time t. Let

$$\tau = \omega t,$$

where ω is the non-linear frequency and will be determined later. Then Eq. (4) becomes

$$\omega^2 x'' + c_1 x + c_3 x^3 = \varepsilon f(x, \omega x'), \tag{5}$$

in which primes denote derivatives with respect to the new variable τ. Let

$$\omega = \sum_{n=0}^{\infty} \varepsilon^n \omega_n; \tag{6}$$

then define a new parameter

$$\alpha = \frac{\varepsilon}{\omega_0 + \varepsilon} \tag{7}$$

such that

$$\varepsilon = \frac{\omega_0 \alpha}{1 - \alpha}. \tag{8}$$

Then $\alpha \in [0, 1)$, when $\varepsilon \in [0, +\infty)$, so ε can be written as

$$\varepsilon = \omega_0 \alpha (1 + \alpha + \alpha^2 + \alpha^3 + \cdots). \tag{9}$$

Substituting Eq. (9) into Eq. (6) we can get that

$$\omega = \omega_0 + \alpha\omega_0\omega_1 + \alpha^2\omega_0(\omega_1 + \omega_0\omega_2) + \alpha^3\omega_0(\omega_1 + 2\omega_0\omega_2 + \omega_0^2\omega_3) + \cdots. \tag{10}$$

Let

$$x = \sum_{n=0}^{\infty} \alpha^n x_n(\tau), \tag{11}$$

expanding $f(x, \omega x')$ in power series of ε, substituting Eqs. (8), (10) and (11) into Eq. (5) and equating the coefficients of powers of α yields the following equations:

$$\alpha^0: \quad \omega_0^2 x_0'' + c_1 x_0 + c_3 x_0^3 = 0, \tag{12}$$

$$\alpha^1: \quad \omega_0^2 x_1'' + (c_1 + 3c_3 x_0^2) x_1 = \omega_0 f(x_0, \omega_0 x_0') - 2\omega_0^2 \omega_1 x_0'', \tag{13}$$

$$\begin{aligned} \alpha^2: \quad \omega^2 x_2'' + (c_1 + c_3 x_0^2) x_2 &= \omega_0 f(x_0, \omega_0 x_0') + \omega_0 f_x'(x_0, \omega_0 x_0') x_1 + \omega_0^2 f_{\dot{x}}'(x_0, \omega_0 x_0')(\omega_1 x_0' + x_1') \\ &\quad - \omega_0^2(\omega_1^2 + 2\omega_1 + 2\omega_0\omega_2) x_0'' - 2\omega_0^2 \omega_1 x_1'' - 3c_3 x_0 x_1^2, \end{aligned} \tag{14}$$

in which $f_x' = \mathrm{d}f/\mathrm{d}x,\ f_{\dot{x}}' = \mathrm{d}f/\mathrm{d}\dot{x}$.

Eq. (12) has an exact analytical solution which can be denoted by

$$x_0(\tau) = A_0 ep(\tau, k), \tag{15}$$

where $ep(\tau, k)$ is one of convenient Jacobian elliptic functions $sn(\tau, k), cn(\tau, k)$ and $dn(\tau, k)$ according to the type of Eq. (12) which is determined by the value of c_1 and c_3 (see Ref. [9]). A_0 and k are constants to be determined later.

Multiplying both sides of Eq. (13) by x_0' and then integrating the equation, we obtain

$$\omega_0^2[x_0'x_1' - x_0''x_1]|_0^\tau + \int_0^\tau [\omega_0^2 x_0''' + c_1 x_0' + 3c_3 x_0^2 x_0']x_1\,\mathrm{d}\tau = \omega_0\left[\int_0^\tau f(x_0, \omega_0 x_0')x_0'\,\mathrm{d}\tau - \omega_0\omega_1 x_0'^2|_0^\tau\right]. \quad (16)$$

Differentiating Eq. (12) with respect to τ leads to

$$\omega_0^2 x_0''' + c_1 x_0' + 3c_3 x_0^2 x_0' = 0. \quad (17)$$

Note that x_0 is a periodic function with period T (T is 4 K for $sn\tau$ and $cn\tau$ or 2 K for $\mathrm{d}n\,\tau$, K is the first kind complete elliptic integral). x_0', x_0'' are also periodic functions with period 4 K. According to our assumption, x_1 is also a periodic function with the period 4 K. Then by letting $\tau = 4$ K in Eq. (16), we have

$$\int_0^{4K} f(x_0, \omega_0 x_0')x_0'\,\mathrm{d}\tau = 0. \quad (18)$$

A_0 can be determined from Eq. (18).

It can be seen from Eq. (17) that x_0' is a solution of the homogeneous part of Eq. (13). Therefore, the particular solution of Eq. (13) can be expressed by the following equation according to the theory of differential equations:

$$x_1 = x_0' \int \frac{1}{x_0'^2}\left\{\int \frac{x_0'}{\omega_0^2}[\omega_0(f(x_0, \omega_0 x_0') - 2\omega_0\omega_1 x_0'')]\,\mathrm{d}\tau\right\}\mathrm{d}\tau. \quad (19)$$

Here we ignore the initial conditions and the homogeneous solution in x_1 since we concerned with steady state solutions, in which the responses are frequently independent of the initial conditions (see Ref. [11]). Note that

$$x_0' \int \frac{1}{x_0'^2}\left[\int 2\omega_1 x_0' x_0''\right]\mathrm{d}\tau = \omega_1 x_0' \tau, \quad (20)$$

$x_0'\tau$ is a secular term. It tends to infinity as $\tau \to \infty$. In order to avoid this secular term, ω_1 is chosen to eliminate the coefficient of x_0'' in the bracket on the right hand side of Eq. (19). If $f(x_0, \omega_0, x_0')$ does not contain the term x_0'' explicitly or implicitly then $\omega_1 = 0$.

One can continue the perturbation procedure to determine the next order solution x_2 and ω_2. In this paper we only compute x_1 and ω_1 because the computation practices show that the solution to the order αx_1 is accurate enough.

3. A study of the three types of generalized Van der Pol oscillator

As an application of the modified elliptic L–P method, we study the limit cycles of the generalized Van der Pol oscillator

$$\ddot{x} + c_1 x + c_3 x^3 = \varepsilon(c_0 - c_2 x^2)\dot{x}. \quad (21)$$

Here

$$f(x_0, \omega_0 x_0') = (c_0 - c_2 x_0^2)\omega_0 x_0'. \tag{22}$$

Obviously, $f(x_0, \omega_0 x_0')$ does not contain x_0''. According to the discussion in above section, we have $\omega_1 = 0$. Therefore,

$$x_1 = x_0' \int \frac{1}{x_0'^2}\left[\int \frac{x_0'}{\omega_0} f(x_0, \omega_0 x_0')\,\mathrm{d}\tau\right]\mathrm{d}\tau. \tag{23}$$

When $\varepsilon = 0$, the so-called generating equation can be divided into three types which have different fundamental generating functions according to the value of c_1 and c_3 (see Ref. [9]).

3.1. Oscillator type I: $c_1 > 0$, $c_3 > 0$

For this type of oscillator, the generating function is taken as

$$ep(\tau, k) = cn(\tau, k) \tag{24}$$

so that the solution of Eq. (12) is

$$x_0 = A_0 cn(\tau, k), \tag{25}$$

$$\omega_0^2 = c_1 + c_3 A_0^2, \tag{26}$$

$$k^2 = \frac{c_3 A_0^2}{2\omega_0^2}. \tag{27}$$

From Eq. (18), we get that A_0 can be determined by the following equation:

$$c_0 I_{11}^K - c_2 A_0^2 I_{12}^K = 0, \tag{28}$$

in which

$$I_{11}^K = \int_0^{4K} sn^2\tau\,\mathrm{d}n^2\tau\,\mathrm{d}\tau = \frac{4}{3k^2}[k'^2 K + (2k^2 - 1)E], \tag{29}$$

$$I_{12}^K = \int_0^{4K} sn^2\tau\, cn^2\tau\,\mathrm{d}n^2\,\tau\,\mathrm{d}\tau = \frac{4}{15k^4}[k'^2(k^2-2)K + (2k^4 + k'^2)E], \tag{30}$$

where E is the second kind complete elliptic integral.

Take the same procedure as in Ref. [10], we have

$$\begin{aligned} I_1 &= \int f(x_0, \omega_0 x_0') x_0'\,\mathrm{d}\tau = \int (c_0 - c_2 A_0^2 cn^2\tau)\omega_0 A_0^2 sn^2\tau\,\mathrm{d}n^2\,\tau\,\mathrm{d}\tau \\ &= \omega_0 A_0^2 (C_{11} sn\tau\, cn\tau + C_{12} sn\tau^3 cn\tau + C_{13} sn\tau^5\, cn\tau \\ &\quad + C_{14} sn\tau\, cn\tau\,\mathrm{d}n\,\tau + C_{15} sn\tau^3 cn\tau\,\mathrm{d}n\,\tau + \cdots, \end{aligned} \tag{31}$$

$$x_1 = x_0' \int \frac{1}{\omega_0 x_0'^2} I_1\,\mathrm{d}\tau = x_0' \sum_{j=1}^{5} C_{1j} IS_{1j} + \cdots, \tag{32}$$

$$x_1' = x_0'' \sum_{j=1}^{5} C_{1j} IS_{1j} + x_0' \sum_{j=1}^{5} C_{1j} SC_{1j} + \cdots. \tag{33}$$

The coefficients C_{1j} ($j = 1, \ldots, 5$) are listed in Appendix A. The functions IS_{1j} and SC_{1j} are given in detail in Appendix B.

3.2. Oscillator Type II: $c_1 > 0$, $c_3 < 0$

For this type of oscillator, the generating function is taken as

$$ep(\tau, k) = sn(\tau, k) \tag{34}$$

so that the solution of Eq. (12) is

$$x_0 = A_0 sn(\tau, k), \tag{35}$$

$$\omega_0^2 = c_1 + \tfrac{1}{2} c_3 A_0^2, \tag{36}$$

$$k^2 = -\frac{c_3 A_0^2}{2\omega_0^2}. \tag{37}$$

From Eq. (18), we get that A_0 can be determined by the following equation:

$$c_0 I_{21}^K - c_2 A_0^2 I_{12}^K = 0, \tag{38}$$

in which

$$I_{21}^K = \int_0^{4K} cn^2\tau \, \mathrm{d}n^2\tau \, \mathrm{d}\tau = \frac{4}{3k^2}[(1 + k^2)E - k'^2 K]. \tag{39}$$

Following the procedure of Ref. [10], we have

$$\begin{aligned} I_2 &= \int f(x_0, \omega_0 x_0') x_0' \, \mathrm{d}\tau = \int (c_0 - c_2 A_0^2 sn^2\tau) \omega_0 A_0^2 cn^2\tau \, \mathrm{d}n^2\tau \, \mathrm{d}\tau \\ &= \omega_0 A_0^2 (C_{21} sn\tau \, cn\tau + C_{22} sn\tau^3 cn\tau + C_{23} sn\tau^5 cn\tau \\ &\quad + C_{24} sn\tau \, cn\tau \, \mathrm{d}n\,\tau + C_{25} sn\tau^3 \, cn\tau \, \mathrm{d}n\,\tau + \cdots, \end{aligned} \tag{40}$$

$$x_1 = x_0' \int \frac{1}{\omega_0 x_0'^2} I_2 \, \mathrm{d}\tau = x_0' \sum_{j=1}^{5} C_{2j} IS_{2j} + \cdots, \tag{41}$$

$$x_1' = x_0'' \sum_{j=1}^{5} C_{2j} IS_{2j} + x_0' \sum_{j=1}^{5} C_{2j} SC_{2j} + \cdots. \tag{42}$$

The coefficients C_{2j} ($j = 1, \ldots, 5$) are listed in Appendix A. The functions IS_{2j} and SC_{2j} are given in detail in Appendix B.

3.3. Oscillator type III: $c_1 < 0,\ c_3 > 0$

For this type of oscillator, the generating function is taken as

$$ep(\tau, k) = \mathrm{d}n(\tau, k) \tag{43}$$

so that the solution of Eq. (12) is

$$x_0 = A_0\,\mathrm{d}n(\tau, k), \tag{44}$$

$$\omega_0^2 = \tfrac{1}{2} c_3 A_0^2, \tag{45}$$

$$k^2 = 2\left(1 + \frac{c_1}{c_3 A_0^2}\right). \tag{46}$$

From Eq. (18), we get that A_0 can be determined by the following equation:

$$c_0 I_{31}^K - c_2 A_0^2 I_{12}^K = 0, \tag{47}$$

in which

$$I_{31}^K = \int_0^{4K} sn^2\tau\, cn^2\tau\, \mathrm{d}\tau = \frac{4}{3k^2}[(2-k^2)E - 2k'^2 K]. \tag{48}$$

Following the procedure of Ref. [10], we have

$$\begin{aligned} I_3 &= \int f(x_0, \omega_0 x_0')x_0'\, \mathrm{d}\tau = \int (c_0 - c_2 A_0^2\, \mathrm{d}n^2\tau)\omega_0 A_0^2 k^4 sn^2\tau\, cn^2\tau\, \mathrm{d}\tau \\ &= \omega_0 A_0^2 k^4 (C_{31} sn\tau\, cn\tau + C_{32} sn\tau^3 cn\tau + C_{33} sn\tau^5 cn\tau \\ &\quad + C_{34} sn\tau\, cn\tau\, \mathrm{d}n\tau + C_{35} sn\tau^3 cn\tau\, \mathrm{d}n\tau + \cdots, \end{aligned} \tag{49}$$

$$x_1 = x_0' \int \frac{1}{\omega_0 x_0'^2} I_3\, \mathrm{d}\tau = x_0' \sum_{j=1}^{5} C_{3j} IS_{3j} + \cdots, \tag{50}$$

$$x_1' = x_0'' \sum_{j=1}^{5} C_{3j} IS_{3j} + x_0' \sum_{j=1}^{5} C_{3j} SC_{3j} + \cdots. \tag{51}$$

The coefficients C_{3j} ($j = 1, \ldots, 5$) are listed in Appendix A. The functions IS_{3j} and SC_{3j} are given in detail in Appendix B.

According to the discussion in Ref. [9], the value of A_0 determined from Eq. (47) must satisfy the condition

$$A_0^2 < -\frac{2c_1}{c_3}, \tag{52}$$

otherwise, there is no solution $x_0 = A_0\,\mathrm{d}n(\tau)$. However, in this case Eq. (12) may have another solution $x_0 = A_0 cn(\tau)$ if A_0 satisfy the condition

$$A_0^2 > -\frac{2c_1}{c_3}. \tag{53}$$

Of course, A_0 in this course is determined from Eq. (38) and x_1, x_1' are determined from Eqs. (41) and (42).

4. Examples

Example 1. Consider the equation

$$\ddot{x} + x^3 = \varepsilon(1.2 - x^2)\dot{x}. \tag{54}$$

This is a special case of the type I oscillator with $c_1 = 0,\ c_3 = 1,\ c_0 = 1.2$ and $c_2 = 1$ so the first perturbation solution is $x_0 = A_0 cn(\tau, k)$. The limit cycle phase portraits for the case $\varepsilon = 0.1,\ \varepsilon = 0.5,\ \varepsilon = 1.0$ are shown in Fig. 1.

Example 2. Consider the equation

$$\ddot{x} + 2x - 0.45x^3 = \varepsilon(1 - x^2)\,\dot{x}\,. \tag{55}$$

This is a type II oscillator with $c_1 = 2,\ c_3 = -0.45,\ c_0 = 1$ and $c_2 = 1$ so the first perturbation solution is $x_0 = A_0 sn(\tau, k)$. The limit cycle phase portraits for the case $\varepsilon = 0.1, \varepsilon = 0.4, \varepsilon = 0.8$ are shown in Fig. 2.

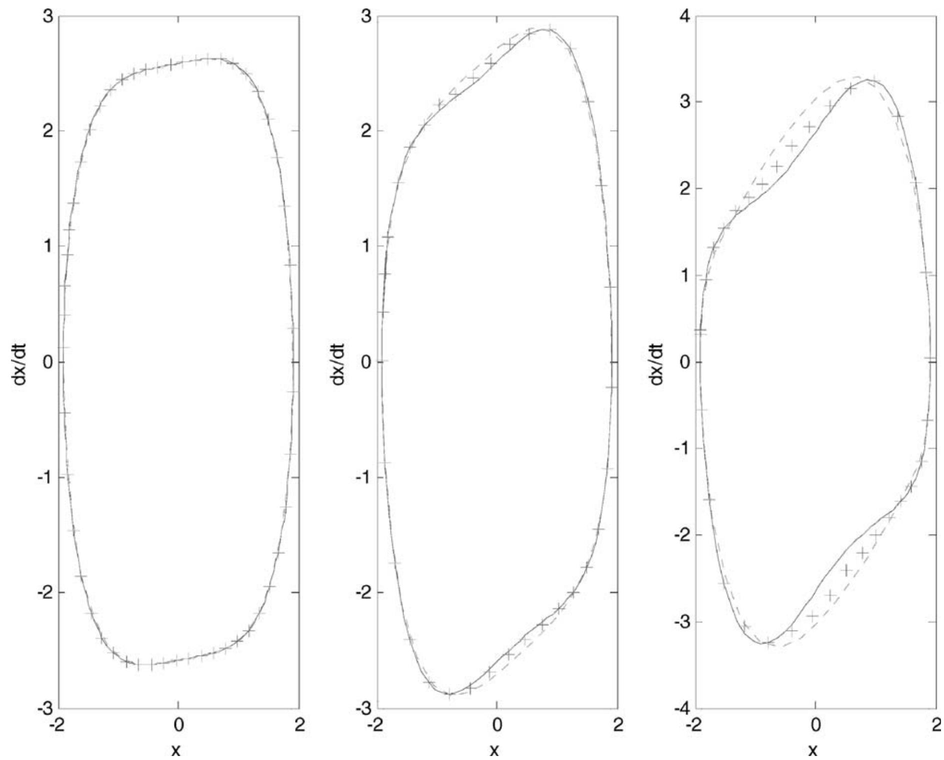

Fig. 1. Limit cycles of Eq. (54). —, R–K method; +, present method; - - -, E-L–P method. (a) $\varepsilon = 0.1$; (b) $\varepsilon = 0.5$; (c) $\varepsilon = 1.0$.

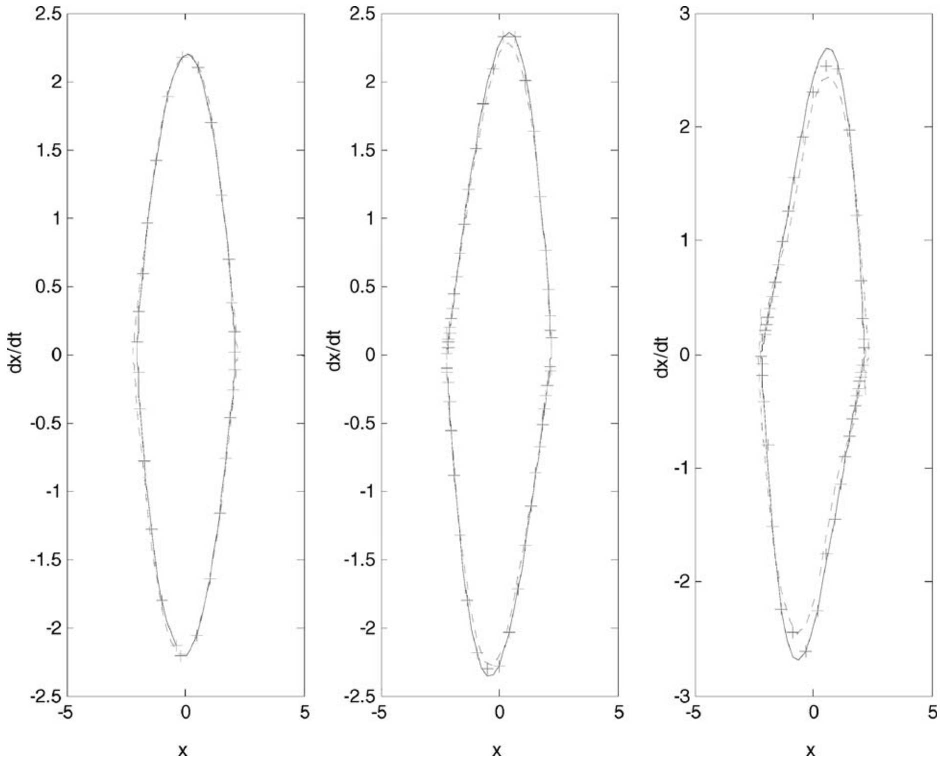

Fig. 2. Limit cycles of Eq. (55). —, R–K method; +, present method; - - -, E-L–P method. (a) $\varepsilon = 0.1$; (b) $\varepsilon = 0.4$; (c) $\varepsilon = 0.8$.

It can be seen from Figs. 1 and 2 that when ε is small both the results of our present method and the elliptic L–P method are nearly identical with the R–K solutions. As ε increases, the present method agrees with the R–K method better than the elliptic L–P method does. The reason is that in the present method we use a new parameter $\alpha = \alpha(\varepsilon)$ instead of the original parameter ε thus the error is decreased.

5. Conclusions

1. A modified elliptic Lindstedt-Poincaré method is presented. It is an efficient method to determine the limit cycles shapes of certain strongly non-linear oscillators in which the periodic solution of the generating equation can be expressed by Jacobian elliptic functions exactly.
2. The comparison between the elliptic Lindstedt–Poincaré method and present method shows that the approximations of both methods are in good agreement with those of the R–K method when ε is small. When ε becomes large, the present method improves the accuracy of the approximations.

Appendix A

1. The coefficients c_{Ej} and $c_{\tau j}$ $(j = 0, 1, 2, 3)$ occurring in the following C_{ij} $(i = 1, 2, 3; j = 0, 1, 2, 3)$ are

$$c_{E0} = 1 - \tfrac{1}{4}k^2 - \tfrac{3}{64}k^4 - \tfrac{5}{256}k^6 + \cdots, \quad c_{E1} = \tfrac{1}{4}k^2 + \tfrac{3}{64}k^4 + \tfrac{5}{356}k^6 + \cdots,$$

$$c_{E2} = \tfrac{1}{32}k^4 + \tfrac{5}{384}k^6 + \cdots, \quad c_{E3} = \tfrac{1}{96}k^6 + \cdots,$$

$$c_{\tau 0} = 1 + \tfrac{1}{4}k^2 + \tfrac{9}{64}k^4 + \tfrac{25}{256}k^6 + \cdots, \quad c_{\tau 1} = -\tfrac{1}{4}k^2 - \tfrac{9}{64}k^4 - \tfrac{25}{256}k^6 + \cdots,$$

$$c_{\tau 2} = -\tfrac{3}{32}k^4 - \tfrac{25}{384}k^6 + \cdots, \quad c_{\tau 3} = -\tfrac{5}{96}k^6 + \cdots.$$

2. The coefficients C_{1j} $(j = 0, 1, \ldots, 5)$ occurring in Eqs. (31)–(33) are

$$C_{1j} = \frac{c_0}{3k^2}[(2k^2 - 1)c_{Ej} + k'^2 c_{\tau j}] - A_0^2 \frac{c_2}{15k^4}[2(k^4 + k'^2)c_{Ej} + k'^2(k^2 - 2)c_{\tau j}], \quad j = 0, 1, 2, 3,$$

$$C_{14} = -\frac{c_0}{3} + \frac{c_2}{15k^2}A_0^2(1 + k^2), \quad C_{15} = -\frac{c_2}{5}A_0^2.$$

3. The coefficients C_{2j} $(j = 0, 1, \ldots, 5)$ occurring in Eqs. (40)–(42) are

$$C_{2j} = \frac{c_0}{3k^2}[(1 + k^2)c_{Ej} - k'^2 c_{\tau j}] - A_0^2 \frac{c_2}{15k^4}[2(k^4 + k'^2)c_{Ej} + k'^2(k^2 - 2)c_{\tau j}], \quad j = 0, 1, 2, 3$$

$$C_{24} = \frac{c_0}{3} + \frac{c_2}{15k^2}A_0^2(1 + k^2), \quad C_{25} = -\frac{c_2}{5}A_0^2.$$

4. The coefficients C_{2j} $(j = 0, 1, \ldots, 5)$ occurring in Eqs. (49)–(51) are

$$C_{3j} = \frac{c_0}{3k^2}[(2 - k^2)c_{Ej} - 2k'^2 c_{\tau j}] - A_0^2 \frac{c_2}{15k^4}[2(k^4 + k'^2)c_{Ej} + k'^2(k^2 - 2)c_{\tau j}], \quad j = 0, 1, 2, 3,$$

$$C_{34} = -\frac{c_0}{3} + \frac{c_2}{15k^2}A_0^2(1 + k^2), \quad C_{35} = -\frac{c_2}{5}A_0^2.$$

Appendix B

1. The functions IS_{1j} and SC_{1j} occurring in Eqs. (31)–(33) are

$$SC_{11} = \frac{cn\tau}{sn\tau\,\mathrm{d}n^2\,\tau}, \quad IS_{11} = \tfrac{1}{2}\ln\frac{1 - \mathrm{d}n\,\tau}{1 + \mathrm{d}n\,\tau} + \frac{1}{\mathrm{d}n\,\tau},$$

$$SC_{12} = \frac{sn\tau\; cn\tau}{\mathrm{d}n^2\,\tau}, \quad IS_{12} = \frac{1}{k^2\,\mathrm{d}n\,\tau},$$

$$SC_{13} = \frac{sn^3\tau\; cn\tau}{\mathrm{d}n^2\,\tau}, \quad IS_{13} = \frac{1}{k^4}\left[\frac{1}{\mathrm{d}n\,\tau} + \mathrm{d}n\,\tau\right],$$

$$SC_{14} = \frac{cn\tau}{sn\tau\,\mathrm{d}n\,\tau}, \quad IS_{14} = \ln\frac{sn\tau}{\mathrm{d}n\,\tau},$$

$$SC_{15} = \frac{sn\tau\;cn\tau}{\mathrm{d}n\,\tau}, \quad IS_{15} = -\frac{1}{k^2}\ln(\mathrm{d}n\,\tau).$$

2. The functions IS_{2j} and SC_{2j} occurring in Eqs. (40)–(42) are

$$SC_{21} = \frac{sn\tau}{cn\tau\,\mathrm{d}n^2\,\tau}, \quad IS_{21} = \frac{1}{2k'^2}\left[\frac{1}{k'}\ln\frac{\mathrm{d}n\,\tau+k'}{\mathrm{d}n\,\tau-k'} - \frac{2}{\mathrm{d}n\,\tau}\right],$$

$$SC_{22} = \frac{sn^3\tau}{cn\tau\,\mathrm{d}n^2\tau}, \quad IS_{22} = \frac{1}{2k'^2}\left[\frac{1}{k'}\ln\frac{\mathrm{d}n\,\tau+k'}{\mathrm{d}n\,\tau-k'} - \frac{2}{k^2\,\mathrm{d}n\,\tau}\right],$$

$$SC_{23} = \frac{sn^5\tau}{cn\tau\,\mathrm{d}n^2\tau}, \quad IS_{23} = \frac{1}{2k'^2}\left[\frac{1}{k'},\ln\frac{\mathrm{d}n\,\tau+k'}{\mathrm{d}n\,\tau-k'} - \frac{2}{k^4\,\mathrm{d}n\,\tau} - \frac{2k'^2}{k^4}\mathrm{d}n\,\tau\right],$$

$$SC_{24} = \frac{sn\tau}{cn\tau\,\mathrm{d}n\,\tau}, \quad IS_{24} = \frac{1}{k'^2}\ln\frac{\mathrm{d}n\,\tau}{cn\tau},$$

$$SC_{25} = \frac{sn^3\tau}{cn\tau\,\mathrm{d}n\,\tau}, \quad IS_{25} = \frac{1}{k'^2}\left[\frac{1}{k^2}\ln(\mathrm{d}n\,\tau) - \ln(cn\tau)\right].$$

3. The functions IS_{3j} and SC_{3j} occurring in Eqs. (49)–(51) are

$$SC_{31} = \frac{1}{sn\tau\;cn\tau}, \quad IS_{31} = \ln\frac{1-\mathrm{d}n\,\tau}{sn\tau} + \frac{1}{k'}\ln\frac{\mathrm{d}n\tau+k'}{cn\tau},$$

$$SC_{32} = \frac{sn\tau}{cn\tau}, \quad IS_{32} = \frac{1}{2k'}\ln\frac{\mathrm{d}n\,\tau+k'}{\mathrm{d}n\,\tau-k'},$$

$$SC_{33} = \frac{sn^3\tau}{cn\tau}, \quad IS_{33} = \frac{1}{k^2}\mathrm{d}n\,\tau + \frac{1}{2k'}\ln\frac{\mathrm{d}n\,\tau+k'}{\mathrm{d}n\,\tau-k'},$$

$$SC_{34} = \frac{\mathrm{d}n\,\tau}{sn\tau\;cn\tau}, \quad IS_{34} = \ln\frac{sn\tau}{cn\tau},$$

$$SC_{35} = \frac{sn\tau\,\mathrm{d}n\,\tau}{cn\tau}, \quad IS_{35} = -\ln(cn\tau).$$

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